Power Series

Power Series :- A series of the type $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is called a power series about the point x_0.

In general, a series of the type $\sum_{n=0}^{\infty} a_n x^n$ is called a Power series about the point $x_0 = 0$.

Radius of Convergence (ROC) :- If $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is a power series. Then its radius of convergence about point x_0 is defined as.

$$R = \frac{1}{l} \quad \text{where} \quad l = \limsup_{n\to\infty} |a_n|^{1/n}$$

OR:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \& \quad l = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

* If R be the ROC of the power series $\sum_{n=0}^{\infty} a_n x^n$ then ROC of $\sum_{n=0}^{\infty} a_n x^{\alpha n}$ is $(R)^{1/\alpha}$

Que: Find ROC of power series $\sum_{n=1}^{\infty} \frac{n}{n+1} x^n$ is.

a) 1 b) 2 c) 3 d) 4.

Solⁿ: Here $a_n = \frac{n}{n+1}$

$$R = \lim_{n\to\infty} \left| \frac{\frac{n}{n+1}}{\frac{n+1}{n+2}} \right| = \lim_{n\to\infty} \left| \frac{n(n+2)}{(n+1)^2} \right| = \lim_{n\to\infty} \frac{n^2\left[1+\frac{2}{n}\right]}{n^2\left[1+\frac{1}{n}\right]^2}$$

$$R = \lim_{n\to\infty} \frac{\left[1+\frac{2}{n}\right]}{\left[1+\frac{1}{n}\right]^2} = 1.$$

∴ option (a) is true.

OR.

$$R = \frac{1}{l} \quad \text{where } l = \lim_{n\to\infty} \sup |a_n|^{1/n}$$

$$l = \lim_{n\to\infty} \sup \left|\frac{n}{n+1}\right|^{1/n}$$

$$l = \lim_{n\to\infty} \sup \frac{|n|^{1/n}}{(1+n)^{1/n}} = \frac{1}{1} = 1$$

$$\therefore \boxed{R = 1}$$

Que: The radius of convergence of power series $\sum_{n=1}^{\infty} \left(\frac{n+2}{n}\right) x^n$ is ___

a) 2 b) $1/\sqrt{2}$ c) $1/2$ d) $1/e^2$

Sol^n: Here $a_n = \frac{n+2}{n}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{n+2}{n}}{\frac{n+3}{n+1}}\right|^{n^2} = \lim_{n\to\infty} \frac{(n+2)(n+1)}{n(n+1)}$$

$$= \lim_{n\to\infty} \frac{n^2\left[1+\frac{2}{n}\right]\left[1+\frac{1}{n}\right]}{n^2\left[1+\frac{1}{n}\right]} = 1 \quad \text{Ans}$$

Que: Find ROC of $\sum_{n=1}^{\infty} \left(\frac{n+2}{n}\right)^{n^2} x^n$ {[JAM-2020]

a) e b) $1/\sqrt{e}$ c) $1/e$ d) $1/e^2$

Sol^n: Here $a_n = \left(\frac{n+2}{n}\right)^{n^2}$

$$R = \frac{1}{l} \quad \text{where } l = \lim_{n\to\infty} \sup |a_n|^{1/n}$$

$$l = \lim_{n\to\infty} \sup \left|\left(\frac{n+2}{n}\right)^{n^2}\right|^{1/n}$$

$$= \lim_{n\to\infty} \sup \left|\frac{n+2}{n}\right|^{n} = \lim_{n\to\infty} \sup \left|1+\frac{2}{n}\right|^{n}$$

$$l = e^2 \qquad \left\{\left[\text{as } \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a\right]\right.$$

$$\therefore R = 1/e^2 \quad \text{Ans}$$

Sol^n: Here $a_n = \frac{n^3}{4^n}$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{n^3}{4^n}}{\frac{(n+1)^3}{4^{n+1}}} = \lim_{n\to\infty}\frac{\frac{n^3}{4^n}}{\frac{(n+1)^3}{4^n.4}} = \lim_{n\to\infty}\frac{4n^3}{(n+1)^3}$$

$\boxed{R=4}$ But the given series of power $5n$ then ROC of series is $(4)^{1/5} = \sqrt[5]{4}$

$\therefore$ option (b) is true.

OR. $R = \frac{1}{l}$

$$l = \lim_{n\to\infty} \sup\left|\frac{n^3}{4^n}\right|^{1/n} = \lim_{n\to\infty}\left|\frac{n^{3/n}}{4}\right|$$

$$l = \frac{1}{4} \quad \left\{\left[\because \lim_{n\to\infty} n^{1/n} = 1\right]\right.$$

$\therefore R = 4 \Rightarrow$ ROC of series is $\sqrt[5]{4}$.

Que: The 'r' be ROC of power series $\frac{1}{3} + \frac{x}{5} + \frac{x^2}{3^2} + \frac{x^3}{5^2} + \frac{x^4}{3^3} + \frac{x^5}{5^3} + \cdots$ then the value of r^2 is equal to ____ {[JAM-2022].

Sol^n:

$$\left(\frac{1}{3} + \frac{x^2}{3^2} + \frac{x^4}{3^3} + \frac{x^6}{3^4} + \cdots\right) + \left(\frac{x}{5} + \frac{x^3}{5^2} + \frac{x^5}{5^3} + \cdots\right)$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{x^{2n}}{3^{n+1}}}_{\text{ROC}=R_1} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{5^n} x^{2n-1}}_{\text{ROC}=R_2}$$

Now $\sum_{n=1}^{\infty} \frac{1}{5^n} x^{2n-1} = P_2$

$R_2 = \lim_{n\to\infty} \left| \frac{1}{5^n} \times 5^{n+1} \right| = 5$

ROC of P_2 is $R_2 = (5)^{1/2}$

So Overall power series 's ROC will be 'r' = min (R_1, R_2)

$r = \sqrt{3}$

$\therefore \boxed{r^2 = 3}$ Ans

Interval of Convergence :- The set of values of x for which power series is cgt is called IOC or region of conv.

i.e. $|x - x_0| < R \Rightarrow (x_0 - R, x_0 + R)$ is called IOC

Note: (i) If $R = \infty$ then power series converges $\forall x$

(ii) If $R = 0$ then power series converges for only $x = 0$

(iii) If $0 < R < \infty$ then power series converges $\forall |x| < R$ or $|x - x_0| < R$ and divg $|x| > R$ or $|x - x_0| > R$

(iv) If $|x - x_0| = R$ then power series may or may not converges.

Que: Find IOC of power series $\sum_{n=1}^{\infty} \frac{3}{n 4^n} (x+1)^n$ is

a) $-4 \le x < 2$ b) $-4 \le x \le 2$ c) $-5 \le x < 3$ d) $-5 \le x \le 3$

Soln: $a_n = \frac{3}{n 4^n}$

$l = \lim_{n\to\infty} \sup \left| \frac{3}{n 4^n} \right|^{1/n} = \lim_{n\to\infty} \sup \left[\frac{(3)^{1/n}}{(n)^{1/n} \cdot 4} \right] = \frac{1}{4}$

$\boxed{\frac{1}{l} = R = 4} < \infty$ thus IOC will be $|x - (-1)| < 4$

$\Rightarrow |x + 1| < 4$

$\Rightarrow -4 < x + 1 < 4$

$\Rightarrow \boxed{-5 < x < 3}$

Now at $x = -5$ series will be $\sum_{n=1}^{\infty} \frac{3}{n 4^n} (-4)^n = \sum_{n=1}^{\infty} \frac{3}{n} \frac{1}{4^n} (-1)^n 4^n = \sum_{n=1}^{\infty} \frac{3}{n} (-1)^n$

By Leibnitz's test: (i) $a_n > a_{n+1} \Rightarrow \frac{3}{n} > \frac{3}{n+1}$

(ii) $\lim_{n\to\infty} a_n = 0$

Thus the series is cgt at $x=5$.

Now at $x=3$ series will be $\sum_{n=1}^{\infty} \frac{3}{n}$

By p-series test $\sum \frac{1}{n^p}$ is cgt iff $p \geq 2$

$\therefore$ series is not cgt at $x=3$

thus, IOC is $-5 \leq x < 3$. option (c) is true.

Que: Find the IOC of power series $\sum_{n=1}^{\infty} \frac{1}{(-3)^{n+2}} \frac{(4x-12)^n}{n^2+1}$. {[JAM-2017]

a) $\frac{10}{4} \leq x \leq \frac{14}{4}$ b) $\frac{10}{4} \leq x < \frac{14}{4}$ c) $\frac{9}{4} \leq x < \frac{15}{4}$ d) $\frac{9}{4} \leq x \leq \frac{15}{4}$.

Solⁿ:

$$\sum_{n=1}^{\infty} \frac{1}{(-3)^{n+2}} \frac{4^n (x-3)^n}{n^2+1} = \sum_{n=1} \frac{4^n (x-3)^n}{9(-3)^n(n^2+1)}$$

$$\therefore a_n = \frac{1}{9}\left(-\frac{4}{3}\right)^n \cdot \frac{1}{n^2+1}$$

$$l = \lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{1}{9}\left(\frac{-4}{3}\right)^n \frac{1}{n^2+1} \right|^{1/n}$$

$$= \frac{4}{3} \lim_{n\to\infty} \sup \left| \frac{1}{(9)^{1/n}} \frac{1}{(n^2+1)^{1/n}} \right| = \frac{4}{3} \lim_{n\to\infty} \sup \left| \frac{1}{(9)^{1/n}} \frac{1}{n^{2/n}\left(1+\frac{1}{n^2}\right)^{1/n}} \right|$$

as $\lim_{n\to\infty} (9)^{1/n} = 1$

$$\therefore l = \frac{4}{3} \Rightarrow \boxed{R = \frac{3}{4}}$$

IOC $|x-3| < R \Rightarrow |x-3| < \frac{3}{4} \Rightarrow \frac{9}{4} < x < \frac{15}{4}$

Now at $\frac{9}{4} = x$, series will be $\sum_{n=1}^{\infty} \frac{(-3)^n}{(-3)^{n+2}} \cdot \frac{1}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{9(n^2+1)}$ which is cgt.

at $x = \frac{15}{4}$ series will be $\sum \frac{(-1)^n}{9} \cdot \frac{1}{n^2+1}$ which is cgt by Leibnitz's test

$\therefore$ IOC will be $\frac{9}{4} \leq x \leq \frac{15}{4}$

So, option (D) is correct.

Note: (i) If a power series $\sum a_n x^n$ converges for $x = \alpha$ then it is absolutely convergent for every $x = x_1$ when $|x_1| < |\alpha|$.

(ii) If a power series $\sum a_n x^n$ diverges for $x = x'$ then it is a div for every $x = x''$ when $|x''| > |x'|$.

(iii) If $\sum a_n (x - x_0)^n$ is cgt for $x = \alpha$ then it is absolutely cgt $\forall x$ s.t. $|x - x_0| < |\alpha - x_0|$

2: Let (a_n) be a sequence of Real numbers s.t. the series $\sum_{n=0}^{\infty} a_n (x-2)$ converges at $x = -5$, then ~~this~~ series also converges at. } [JAM-20

a) $x = -6$, b) $= 12$ c) $x = 9$ d) $x = 5$

: as $x = -5 = \alpha$

$\therefore |x - x_0| \ngtr |\alpha - x_0|$

$\Rightarrow |x - 2| > |-5-2|$

$\Rightarrow |x-2| > |-7|$

$\Rightarrow |x-2| > 7$

$\Rightarrow \boxed{-5 < x < 9}$ cgt. only $x = 5$ is $\in (-5, 9)$

$\therefore$ option (d) is true.