"Set - Theory"

\# **Sets** :- A set is a well defined collection of distinct objects. By well defined means, there should be no any confusion regarding the inclusion or exclusion of an element of the set.

- The objects of a sets are called as Element of a set or Member of a set.
- Sets are usually denoted by Capital letters, and elements of a set are usally denoted by small letters.
- $a \in X$ which means 'a' is an element of set 'X'.

\# **Subset** :- Let A and B be any two sets, we say that A is subset of B if every element of A is also an element of set B.
i.e. $A \subseteq B$

- Empty set is always a subset of every set.
- If $A \subseteq B$ and $A \neq B$ then $A \subsetneq B$ or $A \subset B$ "A is proper subset of B".

\# **Equality of two sets** :- Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$. If every element of A belongs to B and every element of B belongs to A.

Symbolically, $A = B$ iff $\forall a \in A \Rightarrow a \in B$.

\# **Power Set** :- Let A be any set, then the collection of all the subsets of A is called the power set of A and denoted by $P(A)$.

$\therefore P(A) = \{ X \mid X \subseteq A \}$

eg. (i) $A = \{\}$ or ϕ

$P(A) = \{\phi\}$

(ii) $A = \{\phi\}$

$P(A) = \{\phi, \{\phi\}\}$

(iii) $A = \{\phi, \{\phi\}\}$

$P(A) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

(iv) $A = \{1, 2, \{3,4\}, \{2\}\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{\{3,4\}\}, \{\{2\}\}, \{1,2\}, \{1, \{3,4\}\}, \{1, \{2\}\}, \{2, \{3,4\}\}, \{2, \{2\}\}, \{\{3,4\}, \{2\}\}, \{1,2,\{3,4\}\}, \{1,2,\{2\}\}, \{1,2,\{3,4\},\{2\}\}\}$.

note: $\phi \notin A$, $3 \notin A$, $\phi \subseteq A$, $\{3,4\} \in A$, $\{3,4\} \notin A$.

Number of subset of a Set = Number of element in power set of a set

$$= 2^n = (1+1)^n$$

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$$

for $n \in \mathbb{N}$; $1 \le r \le n$

${}^nC_r = \#$ (r-elemented subset of a set that has exactly n elements)

Q. Let A be a set having exactly $2n+1$ elements where $(n \ge 1)$ then the number of subsets of A having more than n elements is?

a) 2^{2n} b) 2^{2n-1} c) 2^{n+1} d) 2^{2n+1}.

Sol^n:-

$$|A| = 2n+1$$

$$|P(A)| = 2^{2n+1} = {}^{2n+1}C_0 + {}^{2n+1}C_1 + \cdots + {}^{2n+1}C_n + \cdots + {}^{2n+1}C_{2n+1}$$

$$2^{2n+1} = 2\left[{}^{2n+1}C_n + \cdots + {}^{2n+1}C_{2n+1}\right]$$

$$\Rightarrow {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \cdots + {}^{2n+1}C_{2n+1} = \frac{2^{2n+1}}{2}$$

$$\Rightarrow {}^{2n+1}C_r + {}^{2n+1}C_{n+1} + \cdots + {}^{2n+1}C_{2n+1} = \boxed{2^{2n}}.$$

Operation on Sets :-

(i) **Union** → The union of sets A and B denoted by $A \cup B$, is defined as the set of those elements which either belongs to A or to B.

Symbollically, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

(ii) **Intersection** → The intersection of two sets A and B denoted by $A \cap B$, is defined as the set of those elements which belongs to both A and B.

Symbollically, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Difference of two sets :- Let A and B be any two sets then the difference of A and B is denoted by A−B and it is defined as the set which contains only those elements of A which do not belongs to B.

symbolically, $A-B = \{x : x \in A \text{ and } x \notin B\}$

and $B-A = \{x \mid x \in B \text{ and } x \notin A\}$

Symmetric difference :- Let A and B any two sets then the symmetric difference of A and B is denoted by $A \Delta B$ $(A \oplus B)$ and it is defined as the set

$$A \Delta B = \{x : x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\}$$

$$\Rightarrow A \Delta B = (A \cup B) - (A \cap B)$$

$$= (A-B) \cup (B-A).$$

Cartesian Product :- Let A and B any two sets. The Cartesian pro[duct] of A and B is denoted by $A \times B$ and it is defined as th[e] set consisting ordered pairs (a,b) where $a \in A$ and $b \in B$.

Symbollically, $A \times B = \{(a,b) \mid a \in A, b \in B\}$.

- Cartesian Product is also called as Direct Product.

eg. $A = \{1,2\}$, $B = \{x,y\}$

$A \times B = \{(1,x), (1,y), (2,x), (2,y)\}$

$B \times A = \{(x,1), (y,1), (x,2), (y,2)\}$

- $(a,b) = (c,d)$

$\iff$ $a = c$ and $b = d$.

- $|A| = m$, $|B| = n$

$A = \{a_1, a_2, \ldots, a_m\}$

$B = \{b_1, b_2, \ldots b_n\}$

$|A \times B| = m \times n = |A| \times |B|$

$\Rightarrow |A \times B| = |A| \times |B|$.

* Properties of Cartesian Product :-

(i) In general $A \times B \neq B \times A$. infact $A \times B = B \times A$ iff either $A = [B]$ or one of the A or B is empty.

(ii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(iii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(iv) $(A - B) \times C = (A \times C) - (B \times C)$

(v) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Q. $|A \cap B| = 50$ then find $|(A \times B) \cap (B \times A)|$

a) 50 b) 50^2 c) 2^{50} d) 100

Solⁿ: $|(A \times B) \cap (B \times A)| = |(A \cap B) \times (B \cap A)| = (50 \times 50) = \boxed{50^2}$

Q. Let $W = \{(a_1, a_2, \ldots a_{10}) : a_1, a_2, \ldots, a_{10} \in \{1,2,3\},\ a_i + a_{i+1} \text{ is even}\}$ $\forall\, i = 1, 2, \ldots, 9$.
then the number of elements in the set W is :

a) $2^{10} + 1$ b) 2^{10} c) 3^{10} d) 3^{10}

sol^n let $a_1 = 2$.

$W = \{(2, 2, 2, \ldots, 2) : a_i + a_{i+1} \text{ is even}\} = 1$ element

let $a_1 = 1$ or 3

$W = \{(\square, \square, \ldots, \square)\} = 2^{10}$ elements.

$\therefore$ Total no. of elements $= \underline{\underline{2^{10} + 1}}$.

\# **Relations :-** Let A and B any two sets then a relation are from A to B is any subset of $A \times B$.

i.e. Every subset of $A \times B$ is a relation from A to B and conversly.

* let $|A| = m$ and $|B| = n$.
then, Number of relation from A to B = the number of subsets of $(A \times B)$

$= |P(A \times B)| = 2^{|A \times B|} = 2^{|A||B|}$

Number of relation from A to B. $= 2^{mn}$

* **Types of Relations →**

(i) **Empty Relation** :- Let A and B be any two sets then $\phi \subseteq A \times B$
$\Rightarrow \phi$ is relation from A to B and it is called empty relation.

(ii) **Universal Relation** :- Let A and B be any two sets then $A \times B \subseteq A \times B$
$\Rightarrow R = A \times B$ is also a relation from A to B, this is called universal relation.

(iii) **Identity Relation** :- Let A be any set than the identity relation on A is denoted by I_A and it is defined as

$$I_A = \{(a, a) \mid a \in A\}$$

(iv) <u>Reflexive Relation</u>:- A is relation R on set A is said to be reflexive relation if $I_A \subseteq R$

i.e. $(a,a) \in R \;\forall\; a \in A$

$\Rightarrow$ every element of A is related to itself by relation R

e.g. $A = \{1,2,3,4\}$

$R_1 = \{(1,1),(1,3),(2,3),(2,2)\}$

$R_2 = \{(1,1),(1,2),(2,3),(3,2),(2,2),(3,3),(3,4),(4,4)\}$

R_2 is reflexive on A but R_1 is not reflexive.

(v) <u>Irreflexive relation</u>:- A relation R on set A is said to be irreflexive if $I_A \cap R = \phi$.

i.e. $(a,a) \in R \;\forall\; a \in A$.

e.g. $A = \{1,2,3,4\}$

$R_1 = \{(1,1),(1,2),(3,2),(3,4)\}$

$R_2 = \{(1,2),(2,1),(3,2),(3,1)\}$

R_2 is irreflexive on A but R_1 is not irreflexive.

(vi) <u>Symmetric Relation</u>:- A relation R on A is said to be a symmetric relation if whenever $(a,b) \in R$ we must have $(b,a) \in R$.

Symbollically, if $(a,b) \in R \Rightarrow (b,a) \in R$.

e.g. $A = \{1,2,3\}$

$R_1 = \{(1,1),(1,2),(3,1)\}$

$R_2 = \{(1,1),(2,2)\}$

$R_3 = \{(1,3),(3,1),(3,2),(2,3)\}$

R_2 and R_3 are symmetric relation while R_1 is not symmetric relation.

(vii) <u>Asymmetric Relation</u>:- A relation R on A is said to be asymmetric relation if whenever $(a,b) \in R$ then we have $(a,b) \notin$

i.e. $(a,b) \in R \Rightarrow (a,b) \notin R$.

viii) **Antisymmetric Relation:-** A relation R is said to be antisymmetric relation if $(a,b) \in R$ and $(b,a) \in R$. then a and b are equal i.e. (a,b) & $(b,a) \in R$
$\Rightarrow a=b$.

e.g. On set of +ve integer (Z^+) define $aRb \Leftrightarrow$ a divides b.
than aRb and bRa
$\Rightarrow a|b$ and $b|a$
$\Rightarrow a \leq b$ and $b \leq a$
$\Rightarrow \boxed{a=b}$

x) **Equivalence Relation:-** A relation R is said to be equivalence relation if it is reflexive, symmetric and transitive relation.

) **Transitive Relation:-** A relation R on A is said to be transitive relation. if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.
i.e. If $(a,b) \in R$ and $(b,c) \in R$
$\Rightarrow (a,c) \in R$. then R is transitive.

e.x. On $\mathbb{Z}$, define R as $(a,b) \in R$ if $a-b$ is even
let $(a,b) \in R$ & $(b,c) \in R$
$a-b$ is even & $b-c$ is even
$\Rightarrow (a-b)+(b-c)$ is even
$\Rightarrow (a-c)$ is even
$\Rightarrow (a-c) \in R$ thus R is transitive Relation.

. On set of $\mathbb{Z}$ define a relation R as $(a,b) \in R$ if and only if a^2+b is even.
a) R is equivalence b) R is Symmetric but not transitive
c) R is symmetric but not an equivalence d) R is transitive but not equivalence.

Solⁿ:- (i) Reflexive: a^2+a is even
$\Rightarrow a(a+1)$ is even
$\Rightarrow (a,a) \in R \ \forall a \in A$
R is reflexive.

(ii) Let $(a,b) \in R$
$\Rightarrow a^2+b$ is even

a^2, b is even
$\Rightarrow a, b^2$ is even
$\Rightarrow a+b^2$ is even

a^2, b is odd
$\Rightarrow a, b^2$ is odd
$\Rightarrow a+b^2$ is odd

$\therefore (b,a) \in R$ thus R is Symmetric.

(iii) Transitivity :- let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow a^2+b$ is even & b^2+c is even

$\Rightarrow a^2+b+b^2+c$ is even

$\Rightarrow a^2+\underbrace{b(b+1)}_{\text{even}}+c$ is even

$\Rightarrow (a^2+c)+\underbrace{b(b+1)}_{\text{even}}$ is even

$\Rightarrow a^2+c$ must be even.

$\Rightarrow (a,c) \in R$ $\therefore$ R is transitive.

Hence, R is Equivalence Relation.

Equivalence Class :- Let R an equivalence class on any non-empty set A is denoted $Cl(a)$ or $[a]$ or $\bar{a}$ is equivalence class of A and it is defined as the set

$$Cl(a) = \{x \in A \mid (x,a) \in R\}$$

e.x. Let $A = \{1,2,3,4,5,6\}$

R be relation s.t. $(a,b) \in R \Leftrightarrow 3$ divides $(a-b)$.

$$Cl(1) = \{x \in A \mid (x,1) \in R\}$$
$$= \{x \in A \mid 3 \text{ divides } (x-1)\}$$
$$= \{1,4\} = Cl(4)$$

$$Cl(2) = \{x \in A \mid (x,2) \in R\} = \{x \in A \mid 3 \text{ divides } (x-2)\}$$
$$= \{2,5\} = Cl(5)$$

$$Cl(3) = \{x \in A \mid (x,3) \in R\} = \{x \in A \mid 3 \text{ divides } (x-3)\}$$
$$= \{3,6\} = Cl(6).$$

Distinct equivalence classes $= Cl(1), Cl(2), Cl(3)$.

or $Cl(4), Cl(5), Cl(6)$

Quotient Set :- Let R be an equivalence relation on A then the Quotient space or set is denoted by A/R and it is the collection of all distinct classes of A.

e.x. $A = \{1,2,3,4,5,6\}$

$Cl(1) = Cl(4)$, $Cl(2) = Cl(5)$, $Cl(3) = Cl(6)$

$A/R = \{(1,4),(2,5),(3,6)\} = \{\bar{1},\bar{2},\bar{3}\} = \{\bar{4},\bar{5},\bar{6}\}$.

Properties of Equivalence Class → Let R be an equivalence relation on $A \neq \phi$.

(i) $\forall a \in A \Rightarrow a \in \bar{a}$

(ii) $\bar{a} = \bar{b} \iff (a,b) \in R$

(iii) $\bar{a} = \bar{b}$ or $\bar{a} \cap \bar{b} = \phi$.

(iv) If $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_n$ are distinct classes of set A.

then $A = \bigcup_{i=1}^{n} \bar{a}_i$

Partition of a Set:- Let X be any non-empty set. A collection T of some subsets of X is said to be a partition of X

if: (i) $\forall A \in T \Rightarrow A \neq \phi$

(ii) If $A, B \in T$ s.t. $A \neq B$ then $A \cap B = \phi$

(iii) $\bigcup_{A \in T} A = X$

e.x. (i) $X = \{1\}$

$\Rightarrow T = \{1\}$

(ii) $X = \{1,2\}$

$T_1 = \{\{1\},\{2\}\}$

$T_2 = \{\{1,2\}\}$

(iii) $X = \{1,2,3\}$

$T_1 = \{\{1\},\{2\},\{3\}\}$

$T_2 = \{\{1\},\{2,3\}\}$

$T_3 = \{\{3\},\{1,2\}\}$

$T_4 = \{\{2\},\{1,3\}\}$

$T_5 = \{\{1,2,3\}\}$

Fundamental theorem of Equivalence Relation:- There are as many as Equivalence relations on a set A as there are partitions of the set A.

i.e. No. of equivalence Relations on set A = No. of partition of set A.

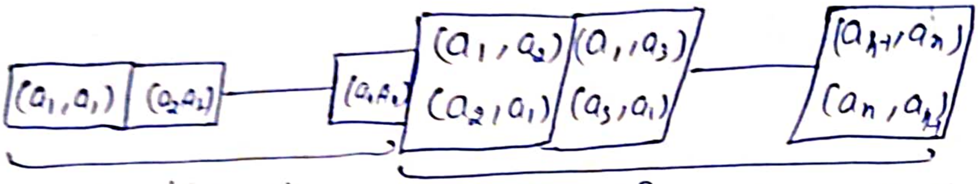
Counting of Relation :- Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$

$$A \times A = \left\{ \begin{matrix} (a_1,a_1), & (a_1,a_2), & (a_1,a_3) & \ldots & (a_1,a_n) \\ (a_2,a_1), & (a_2,a_2) & \ldots & & (a_2,a_n) \\ \vdots & & & & \\ (a_n,a_1), & (a_n,a_2) & \ldots & & (a_n,a_n) \end{matrix} \right\}$$

Total elements $(A \times A) = n^2$

No. of diagonal elements $= n = \{(a_1,a_1),(a_2,a_2),\ldots(a_n,a_n)$

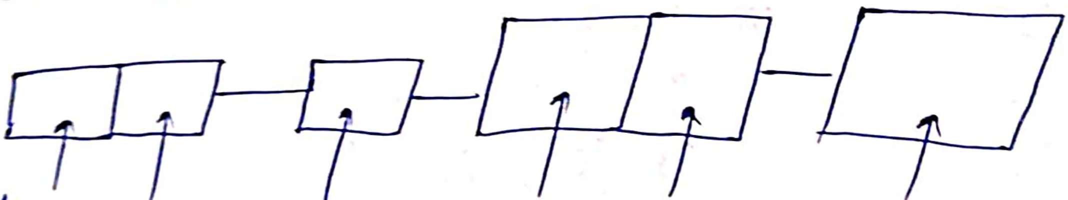
* Identity Relation :- Total no. of identity relation = 1.



n-diagonal element in 'n boxes'

n^2-n elements in "$\left(\frac{n^2-n}{2}\right)$ boxes"

* Reflexive Relation :-



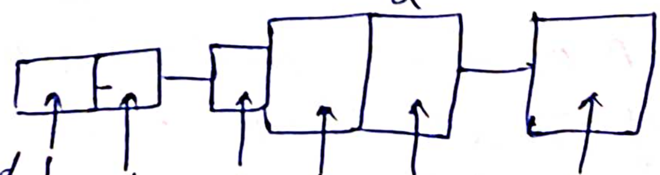
*allowed 1 1 1 4 4 4

Total No. of reflexive Relation $= \underbrace{(1.1.1\ldots1)}_{n\text{ times}}\underbrace{(4.4.4\ldots4)}_{\left(\frac{n^2-n}{2}\right)\text{ times}}$

$$= 4^{\frac{n^2-n}{2}} = \left((2)^2\right)^{\frac{n^2-n}{2}}$$

No. Reflexive Relation $= 2^{n^2-n}$

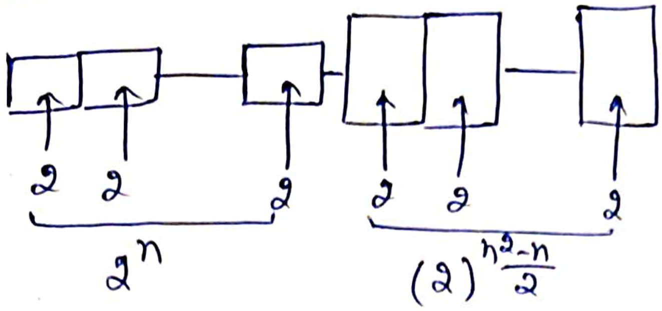
* Irreflexive Relation :-



*not allowed 1 1 1 4 4 4

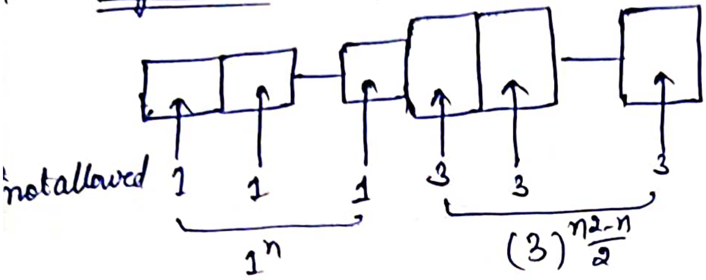
No. of Irreflexive Relation $= 2^{n^2-n}$

* **Symmetric Relation:-**



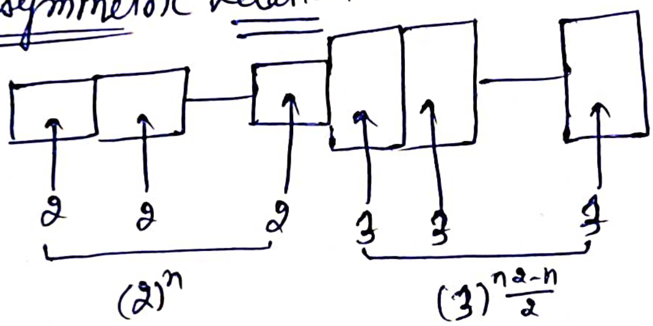
No. of Symmetric Relation $= 2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}} = 2^{\sum n}$

* **Asymmetric Relation:-**



No. of Asymmetric relation $= 3^{\frac{n^2-n}{2}} = 3^{\sum(n-1)}$

* **Antisymmetric Relation:-**



No. of Antisymmetric relation $= 2^n \cdot 3^{\frac{n^2-n}{2}} = 2^n \cdot 3^{\sum(n-1)}$.